

# Geometrical and hydrodynamic aspects of five-dimensional Schwarzschild black hole

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Exploiting the five-dimensional Schwarzschild black hole, we study the geometrical natures of the higher dimensional black hole to yield the (6+1) dimensional global embedding Minkowski space structure. We next obtain the Hawking temperature on this five-dimensional manifold, whose result is different from the four-dimensional one. On the contrary, the radial component of the Einstein equation for the massive particles or photons on the five-dimensional spacetime is shown to have the same form as the four-dimensional black hole one. Moreover, we construct the effective potential on equatorial plane of the restricted three-brane to investigate the behavior of the particles or photons on this restricted brane.

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## I. INTRODUCTION

There have been tremendous progresses in lower dimensional black holes associated with the ten-dimensional string theory [1, 2] since an exact conformal field theory describing a black hole in two-dimensional space-time was proposed [3]. Even though the (2+1) dimensional Bañados-Teitelboim-Zanelli (BTZ) black hole [4–8] is a toy model in some respect, the BTZ black hole has triggered significant interests due to its connections with some string theories [9, 10] on ten-dimensional space-time, its role in microscopic entropy derivations [11, 12] and quantum corrected thermodynamics [13–15]. Specifically, a slightly modified solution of the BTZ black hole yields an solution to the string theory, so-called the black string [16, 17]. Here one notes that this black string solution is in fact only a solution to the lowest order  $\beta$ -function equation to receive quantum corrections [18]. Recently, the Hawking temperatures [19, 20] and Unruh effects [21] of the (2+1) dimensional BTZ black holes have been analyzed [22, 23] and the (2+1) dimensional black strings have been later investigated [24] in the framework of the global embedding Minkowski space (GEMS) structures.

On the other hand, the string singularities, which are the string theory version of the Hawking-Penrose singularities [25], have been applied to the early universe with an arbitrary higher dimensionality [26–28]. In this higher dimensional stringy cosmology, the expansion of the universe has been explained by exploiting Hawking-Penrose type singularity in geodesic surface congruences for the timelike and null strings [26, 27]. Next, the twist and shear have been studied in terms of the expansion of the universe. Moreover, as the early universe evolves with expansion rate, the twist of the stringy congruence decreases exponentially and the initial twist value should be large enough to sustain the rotations of the ensuing universe, while the effects of the shear are negligible to produce the isotropic and homogeneous universe [28]. By exploiting the phantom field, the evolution of cosmology has been also studied in higher dimensional spacetime [29].

Recently, the warp factor has been applied to higher dimensional theories such as the Randall-Sundrum model [30–34] in five-dimensions. The hierarchy problem on the four-brane and five-brane has been studied by attaching a circle and a sphere to the standard three-brane, respectively [35]. The effective masses in their excited spectra on the four- and five-branes has been shown to indicate interesting characteristics associated with the quantization of the compact circular and spherical manifolds, while their lightest effective masses are shown to suppress exponentially with respect to the Planck mass, similar to the standard three-brane case.

In this paper, keeping in mind the gravity and/or cosmology theories related to the ten-dimensional string and in the five-dimensional RS model mentioned above, we will study the higher dimensional general relativity theory by considering the specific five-dimensional Schwarzschild black hole [36, 37].

This paper is organized as follows. In Section II we introduce the five-dimensional Schwarzschild black hole metric to study the GEMS natures of the five-dimensional Schwarzschild black hole, and in Section III we investigate the

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hydrodynamics of the massive particles and photons. In Section IV, we will study an effective potential characteristics on equatorial plane of the restricted three-brane. Section V includes the summaries and discussions. In Appendix A, we study the geometrical properties of five-dimensional Schwarzschild black hole such as the Christoffel symbols associated with the five-acceleration. We also list the Riemann tensors of the five-dimensional Schwarzschild black hole, which will be used in the Einstein equation of interest.

## II. GLOBAL FLAT EMBEDDINGS AND THERMODYNAMICS

After Unruh's work [21], it has been known that a thermal Hawking effect on a curved manifold [19] can be looked at as an Unruh effect in a higher flat dimensional space time. According to the GEMS approach [38–41], several authors [22, 23, 43, 45–47, 52, 53] recently have shown that this approach could yield a unified derivation of temperature for various curved manifolds such as rotating Bañados-Teitelboim-Zanelli (BTZ) [4–8] for instance.

In order to investigate the global flat embedding structure of the five-dimensional Schwarzschild black hole defined on the total manifold  $S^3 \times \mathbf{R}^2$ , we consider the five-metric of the form

$$ds_5^2 = -N^2 dt^2 + N^{-2} dr^2 + r^2 d\Omega_3^2, \quad (2.1)$$

where

$$\begin{aligned} N^2 &= 1 - \frac{\mu_5}{r^2}, \\ d\Omega_3^2 &= d\alpha^2 + \sin^2 \alpha (d\theta^2 + \sin^2 \theta d\phi^2), \end{aligned} \quad (2.2)$$

and  $\Omega_3$  is the solid angle in the three-dimensional compact sphere  $S^3$  whose value is  $2\pi^2$ . Here we have three angles of the three-sphere whose ranges are defined by  $0 \leq \alpha \leq \pi$ ,  $0 \leq \theta \leq \pi$  and  $0 \leq \phi \leq 2\pi$ . In the lapse function  $N$  we have the five-dimensional Schwarzschild radius  $\mu_5$  defined as<sup>1</sup>

$$\mu_5 = \frac{8GM}{3\pi}, \quad (2.3)$$

with the black hole mass  $M$ . Exploiting the Riemann tensors given in (A.2) in Appendix A, one can readily show that the five-dimensional metric (2.1) is the vacuum solution of the Einstein field equations for the exterior region of the five-dimensional black hole. Here, we observe that, in the restriction of  $\alpha = \pi/2$ , the five-dimensional Schwarzschild metric (2.1) reduces to the ordinary four-dimensional Schwarzschild submanifold form [48] with a modified gravitational potential originated from the geometric properties of the five-dimensional Schwarzschild black hole. In fact, in the weak gravity limit, the gravitational potential in the five-dimensional theory is proportional to  $1/r^2$ , different from  $1/r$  of the four-dimensional Newtonian theory. These points will be discussed in more detail in ensuing sections.

Now, we evaluate the horizon radius  $r_H$  by exploiting the vanishing lapse function at  $r = r_H$ , to yield

$$r_H = \mu_5^{1/2}, \quad (2.4)$$

and the surface gravity  $k_H$  is given by

$$k_H = \frac{1}{2} \frac{dN^2}{dr} \Big|_{r=r_H} = \frac{1}{r_H}. \quad (2.5)$$

After some tedious algebra, for the five-dimensional Schwarzschild black hole in the whole region, we obtain the (6+1) global embedding Minkowski space (GEMS) structure

$$ds^2 = -(dz^0)^2 + (dz^1)^2 + (dz^2)^2 + (dz^3)^2 + (dz^4)^2 + (dz^5)^2 + (dz^6)^2 \quad (2.6)$$

with the coordinate transformations

$$z^0 = k_H^{-1} \left( 1 - \frac{\mu_5}{r^2} \right)^{1/2} \sinh k_H t,$$

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<sup>1</sup> The  $d$ -dimensional Schwarzschild radius  $\mu_d$  [37] is given by  $\mu_d = \frac{16GM}{(d-2)\Omega_{d-2}}$ .

$$\begin{aligned}
z^1 &= k_H^{-1} \left(1 - \frac{\mu_5}{r^2}\right)^{1/2} \cosh k_H t, \\
z^2 &= \int dr \left( \frac{r_H^2(r^3 + r_H r^2 + r_H^2 r + r_H^3)}{r^4(r + r_H)} \right)^{1/2}, \\
z^3 &= r \sin \alpha \sin \theta \cos \phi, \\
z^4 &= r \sin \alpha \sin \theta \sin \phi, \\
z^5 &= r \sin \alpha \cos \theta, \\
z^6 &= r \cos \alpha.
\end{aligned} \tag{2.7}$$

Exploiting the Christoffel symbols for the five-dimensional Schwarzschild black hole given in (A.1) of Appendix A, we obtain the five-acceleration

$$a_{d=5} = \frac{r_H^2}{r^2[(r - r_H)(r + r_H)]^{1/2}}, \tag{2.8}$$

and the Hawking temperature  $T_{d=5}^H$  in terms of the above seven-acceleration  $a_{d=7}$  to produce

$$T_{d=5}^H = \frac{a_{d=7}}{2\pi} = \frac{1}{2\pi r_H [1 - (r_H/r)^2]^{1/2}}. \tag{2.9}$$

For the sake of comparison, we note that  $T_{d=4}^H$  in the standard four-dimensional Schwarzschild black hole is given by [53]

$$T_{d=4}^H = \frac{1}{4\pi r_S [1 - (r_S/r)]^{1/2}}, \tag{2.10}$$

where  $r_S = 2GM$ . The difference between the Hawking temperatures in (2.9) and (2.10) originates from the fact that the Newtonian force laws in these two cases are different from each other.

### III. HYDRODYNAMIC PROPERTIES

In this section, we assume theoretically at the moment that the massive particles and photons are moving around the five-dimensional Schwarzschild black hole to investigate their hydrodynamic processes. We consider the five velocity given by  $u^a = dx^a/d\tau$  where one can choose  $\tau$  to be the proper time (affine parameter) for timelike (null) geodesics. From the equations of motion of a massive particle and/or a photon around the five-dimensional Schwarzschild black hole, the massive particles initially at rest and the photons with an initial velocity  $u_\infty^r \approx 1$ , respectively.

Now, the fundamental equations of relativistic fluid dynamics can be obtained from the conservation of particle number and energy-momentum fluxes. In order to derive an equation for the conservation of particle numbers one can use the particle flux four-vector  $nu^a$  to yield

$$\nabla_a(nu^a) = \frac{1}{\sqrt{-g}} \partial_a(\sqrt{-g} nu^a) = 0, \tag{3.1}$$

where  $n$  is the proper number density of particles measured in the rest frame of the fluid of massive particles and photons and  $\nabla_a$  is the covariant derivative in the five-dimensional Schwarzschild curved manifold of interest and the determinant of  $g_{ab}$  is defined by  $g = \det g_{ab}$ . For steady state flow of the perfect fluid of the massive particles and photons, the conservation of energy-momentum fluxes is similarly described by the Einstein equation as below

$$\nabla_b T_a^b = \frac{1}{\sqrt{-g}} \partial_b(\sqrt{-g} T_a^b) = 0, \tag{3.2}$$

where the stress-energy tensor  $T^{ab}$  for perfect fluid is given by

$$T^{ab} = \rho u^a u^b + P(g^{ab} + u^a u^b), \tag{3.3}$$

with  $\rho$  and  $P$  being the proper energy densities and pressures of the massive particles or photons, respectively. In obtaining the first equalities in (3.1) and (3.2), we have used the the Christoffel symbols for the five-dimensional Schwarzschild black hole given in (A.1) of Appendix A.

For the steady state flow of the perfect fluid of the massive particles and photons, the equations (3.1) and (3.2) yield

$$2\pi^2 n u^r r^3 \sin^2 \alpha \sin \theta = A_0, \quad (3.4)$$

$$(P + \rho) u_i u^r r^3 \sin^2 \alpha \sin \theta = A_i, \quad (3.5)$$

where  $A_0$  is the accretion rate of the massive particles or photons, and  $A_i$  ( $i = t, \phi$ ) are the other constants of the motion which can be evaluated at infinity to yield the ratio  $A_\phi/A_t = u_\phi/u_t = 0$ . Combining the equations (3.4) and (3.5), one can derive the relations

$$\frac{(P + \rho)^2}{n^2} \left( \kappa \left( 1 - \frac{\mu_5}{r^2} \right) + u^r u^r \right) = \frac{(P_\infty + \rho_\infty)^2}{n_\infty^2} (\kappa + (1 - \kappa) u_\infty^r u_\infty^r), \quad (3.6)$$

where  $n_\infty$ ,  $P_\infty$  and  $\rho_\infty$  are the particle number density, pressure and internal energy density of the fluid of the massive particles or photons at infinity, respectively. Here we have introduced a new parameter  $\kappa$  defined as

$$\kappa = -g_{ab} u^a u^b = \begin{cases} 1 & \text{for timelike geodesics} \\ 0 & \text{for null geodesics.} \end{cases} \quad (3.7)$$

Next, using the projection operators in (3.2) one can obtain the general relativistic equation on the direction perpendicular to the five-velocity [49]

$$(P + \rho) u^b \nabla_b u_a + (g_{ab} + u_a u_b) \nabla^b P = 0 \quad (3.8)$$

from which, after some algebra, we obtain the radial component of the above equation for the steady state axisymmetric accretion of the massive particles and photons on the five-dimensional Schwarzschild black hole of mass  $M$

$$\frac{1}{2} \frac{d}{dr} (u^r u^r) + \kappa \frac{\mu_5}{r^3} + \frac{1}{P + \rho} \left( u^r u^r + 1 - \frac{\mu_5}{r^2} \right) \frac{dP}{dr} = 0. \quad (3.9)$$

The equation (3.9) is one of the main results, comparable to the spherically symmetric four-dimensional Schwarzschild black hole result [50]

$$\frac{1}{2} \frac{d}{dr} (u^r u^r) + \kappa \frac{M}{r^2} + \frac{1}{P + \rho} \left( u^r u^r + 1 - \frac{2M}{r} \right) \frac{dP}{dr} = 0. \quad (3.10)$$

The Einstein equation (3.2) can be easily rewritten in another covariant form

$$u_a \nabla_b ((P + \rho) u^b) + (P + \rho) u^b \nabla_b u_a + \nabla_a P = 0. \quad (3.11)$$

Multiplying this equation by  $u^a$  one can project it on the direction of the four-velocity to obtain

$$n u^a \left( \nabla_a \frac{P + \rho}{n} - \frac{1}{n} \nabla_a P \right) = 0 \quad (3.12)$$

where the continuity equation (3.1) has been used. The radial component of the equation (3.12) yields<sup>2</sup>

$$\kappa \frac{d\rho}{dr} - \kappa \frac{P + \rho}{n} \frac{dn}{dr} + (\kappa - 1) \frac{dP}{dr} = \frac{\Lambda - \Gamma}{u^r}. \quad (3.13)$$

Here the energy loss  $\Lambda$  and the energy gain  $\Gamma$  are introduced to set the decrease in the entropy of the inflowing massive and/or massless particles equal to difference  $\Lambda - \Gamma$ .

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<sup>2</sup> This generic result in (3.13) is shared by the Kerr black hole [51]. Here we have included the massless photon case, in addition to the massive particle one given in [50].

#### IV. EFFECTIVE POTENTIAL ON EQUATORIAL PLANE OF RESTRICTED THREE-BRANE

In the five-dimensional Schwarzschild metric (2.1), where  $t$  and  $\phi$  are cyclic, we have two Killing vector fields  $\xi_i^a$  ( $i = t, \phi$ ) satisfying the Killing equations

$$\mathcal{L}_{\xi_i} g_{ab} = \nabla_a \xi_{ib} + \nabla_b \xi_{ia} = 0. \quad (4.1)$$

Using the above Killing equations and geodesic ones, one can readily see that  $u^c \nabla_c (g_{ab} \xi_i^a u^b) = 0$  ( $i = t, \phi$ ) to produce the constants of motion  $\epsilon$  and  $l$  corresponding to the Killing vector fields,

$$g_{ab} \xi_t^a u^b = - \left(1 - \frac{\mu_5}{r^2}\right) u^t = -\epsilon, \quad (4.2)$$

$$g_{ab} \xi_\phi^a u^b = r^2 \sin^2 \alpha \sin^2 \theta u^\phi = l, \quad (4.3)$$

where  $\epsilon$  and  $l$  are the conserved energy per unit mass and angular momentum per unit mass, for the massive particles. For the photons, we note that  $\hbar\epsilon$  and  $\hbar l$  are the total energy and the angular momentum of the photons, respectively.

In this section, by taking an ansatz  $\alpha = \pi/2$ , we consider the restricted three-brane times  $\mathbf{R}$  submanifold defined on the hyper-disk  $S^2$  times  $\mathbf{R}^2$ , on which the effective four-dimensional Schwarzschild metric is described. This four-dimensional Schwarzschild black hole defined on the restricted three-brane has a form different from that of the four-metric of the standard Schwarzschild back hole, due to the different type of the modified Newtonian inverse square potential law:

$$ds_4^2 = - \left(1 - \frac{\mu_5}{r^2}\right) dt^2 + \left(1 - \frac{\mu_5}{r^2}\right)^{-2} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2). \quad (4.4)$$

Here we reemphasize that the inverse square potential law holds even on the restricted four-dimensional submanifold. To investigate the global geometrical and topological effects of the five-dimensional Schwarzschild black hole on the effective potentials on the three-brane, we assume that the test particles or photons reside on the equatorial plane with  $\theta = \pi/2$  of the restricted three-brane, so that the four-metric (4.4) can reduce to the three brane where our effective ordinary cosmological world is defined,

$$ds_3^2 = - \left(1 - \frac{\mu_5}{r^2}\right) dt^2 + \left(1 - \frac{\mu_5}{r^2}\right)^{-2} dr^2 + r^2 d\phi^2. \quad (4.5)$$

On this restricted three-brane, the angular momentum  $l$  per unit mass in (4.2) reduces to

$$l = r^2 u^\phi. \quad (4.6)$$

Combining (4.2) and (4.6) with (3.7), we arrive at the equation of motion of the massive particles and photons with the effective potential  $V$ , as follows

$$\frac{1}{2} \dot{r}^2 + V = \frac{1}{2} \epsilon^2, \quad (4.7)$$

where  $\dot{r} = dr/d\tau$  and

$$V = \frac{1}{2} \kappa - \kappa \frac{\mu_5}{2r^2} + \frac{l^2}{2r^2} - \frac{\mu_5 l^2}{2r^4}. \quad (4.8)$$

Here one notes that the second term is the modified Newtonian term due to the higher dimensional total manifold geometry, the third is the corresponding centrifugal term and the last term is the general relativistic term, respectively.

Now, we consider the effective timelike geodesics for the massive particle constrained in the potential (4.8) with the restriction on  $l$ :  $l^2 > \mu_5$ . To investigate the orbit of the particle, we calculate

$$\frac{dV}{dr}(r = R) = 0, \quad (4.9)$$

where  $R$  is the radius of the unstable orbit

$$R = \left( \frac{2\mu_5}{1 - \mu_5/l^2} \right)^{1/2}, \quad (4.10)$$

which is greater than the value of  $r_H$ . Differently from the four-dimensional Schwarzschild black hole case where we have the radii of the stable and unstable orbits, in the five-dimensional Schwarzschild black hole we have only the unstable orbit for the massive particle. To see this fact more carefully, we evaluate the following quantities

$$\frac{d^2V}{dr^2}(r=R) = -\frac{2(l^2 - \mu_5)}{R^4}, \quad (4.11)$$

and

$$V(r=R) = \frac{1}{2} + \frac{(l^2 - \mu_5)^2}{8\mu_5 l^2}, \quad (4.12)$$

which is greater than  $1/2$ . The results (4.11) and (4.12) indicate that the  $V$  at  $r=R$  has the maximum value which is bigger than the asymptotic value  $V = 1/2$  which we can obtain when  $r$  approaches the infinity. We thus have a potential barrier with a peak located at  $r=R$  and the massive particle with  $\epsilon > 1$  thus possesses an unbound orbit. For the case of  $l^2 < \mu_5$ , there exist no extrema of  $V$  and the massive particle is going down along the monotonic potential curve up to the black mass. It is interesting to see that this particle crosses the  $V = 0$  on the circular ring at  $r = r_H$ , the event horizon of the five- or four-dimensional Schwarzschild black hole. After some algebra in our case at hand, we can readily obtain the impact parameter  $b$  as follows,

$$b^2 = \frac{\epsilon^2 - 1}{\epsilon^2} \cdot \frac{R^4}{\mu_5}. \quad (4.13)$$

Next, we consider the null geodesics of photon whose potential is given by inserting  $\kappa = 0$  into (4.8) as follows

$$V = \frac{l^2}{2r^2} - \frac{\mu_5 l^2}{2r^4}. \quad (4.14)$$

The orbit of the photon can be now obtainable by considering the following extrema condition

$$\frac{dV}{dr}(r=R_0) = 0, \quad (4.15)$$

where  $R_0$  is the radius of the unstable orbit

$$R_0 = (2\mu_5)^{1/2}, \quad (4.16)$$

which is also greater than the value of  $r_H$ . Similar to the four-dimensional Schwarzschild black hole, we have the unstable photon orbit radius  $r = R_0$  in the five-dimensional Schwarzschild black hole. Moreover, we can readily check that  $d^2V/dr^2(r=R_0) = -6\mu_5 l^2/R_0^6$  and  $V(r=R_0) = l^2/8\mu_5$  indicating the peak of the potential (4.14) at  $r = R_0$ . Here one notes that on the black hole horizon at  $r = r_H$ , the potential  $V(r=r_H)$  vanishes in both cases of the massive particle and photon. The impact parameter  $b$  for the photon is now given by,

$$b^2 = \frac{R_0^4}{R_0^2 - \mu_5}. \quad (4.17)$$

## V. CONCLUSIONS

We have introduced the five-dimensional Schwarzschild black hole metric to obtain the (6+1) dimensional GEMS structure and the five-acceleration of this higher dimensional black hole. Here one notes that the four-dimensional Schwarzschild black hole has the (5+1) dimensional GEMS structure [52, 53]. We have next evaluated its thermodynamic physical quantity such as the Hawking temperature on this five-dimensional manifold. Here we have noticed that the five-dimensional Hawking temperature has the form different from the four-dimensional one due to the fact that the Newtonian force laws in these two cases are different from each other.

Next, we have investigated the hydrodynamic aspects of the five-dimensional Schwarzschild black hole, by assuming that the massive particles and photons are moving around the black hole. We have obtained the radial component equation for the steady state axisymmetric accretion of the massive particles and photons on the five-dimensional Schwarzschild black hole, and this radial equation has been shown to be different from that of the standard four-dimensional Schwarzschild black hole, only because of the differences in their force laws. We then have found the radial component of the Einstein equation associated with the entropies of the massive particles and photons. Remarkably, this equation has shown to be the same form as the four-dimensional black hole cases.

Finally, we have assumed that the test particles or photons reside on the equatorial plane with  $\theta = \pi/2$  of the restricted three-brane with  $\alpha = \pi/2$ , namely, we could have the three-brane on which the effective ordinary cosmological world is defined. On this restricted brane, we have studied the behaviors of the particles or photons.

## Appendix A: Geometry of five-dimensional Schwarzschild black hole

Using the five-dimensional Schwarzschild black metric (2.1), we list the nonvanishing Christoffel symbols which have been exploited in the evaluation of the five-acceleration  $a_{d=5}$  in (2.8)

$$\begin{aligned}
\Gamma_{tr}^t &= -\Gamma_{rr}^r = \frac{\mu_5}{r(r^2 - \mu_5)}, & \Gamma_{tt}^r &= \frac{\mu_5(r^2 - \mu_5)}{r^5}, \\
\Gamma_{\alpha\alpha}^r &= -\frac{r^2 - \mu_5}{r^4}, & \Gamma_{\theta\theta}^r &= -\frac{(r^2 - \mu_5)\sin^2\alpha}{r}, \\
\Gamma_{\phi\phi}^r &= -\frac{(r^2 - \mu_5)\sin^2\alpha\sin^2\theta}{r}, & \Gamma_{r\alpha}^\alpha &= \Gamma_{r\theta}^\theta = \Gamma_{r\phi}^\phi = \frac{1}{r}, \\
\Gamma_{\theta\theta}^\alpha &= -\sin\alpha\cos\alpha, & \Gamma_{\phi\phi}^\alpha &= -\sin\alpha\cos\alpha\sin^2\theta, \\
\Gamma_{\alpha\theta}^\theta &= \Gamma_{\alpha\phi}^\phi = \cot\alpha, & \Gamma_{\phi\phi}^\theta &= -\sin\theta\cos\theta, \\
\Gamma_{\theta\phi}^\phi &= \cot\theta.
\end{aligned} \tag{A.1}$$

For the sake of completeness, we calculate the nonzero Riemann tensors

$$\begin{aligned}
R_{trtr} &= -\frac{3\mu_5}{r^4}, & R_{t\alpha t\alpha} &= \frac{\mu_5(r^2 - \mu_5)}{r^5}, \\
R_{t\theta t\theta} &= \frac{\mu_5(r^2 - \mu_5)\sin^2\alpha}{r^4}, & R_{t\phi t\phi} &= \frac{\mu_5(r^2 - \mu_5)\sin^2\alpha\sin^2\theta}{r^4}, \\
R_{r\alpha r\alpha} &= -\frac{\mu_5}{r^2 - \mu_5}, & R_{r\theta r\theta} &= -\frac{\mu_5\sin^2\alpha}{r^2 - \mu_5}, \\
R_{r\phi r\phi} &= -\frac{\mu_5\sin^2\alpha\sin^2\theta}{r^2 - \mu_5}, & R_{\alpha\theta\alpha\theta} &= \mu_5\sin^2\alpha, \\
R_{\alpha\phi\alpha\phi} &= \mu_5\sin^2\alpha\sin^2\theta, & R_{\theta\phi\theta\phi} &= \mu_5\sin^4\alpha\sin^2\theta,
\end{aligned} \tag{A.2}$$

which yield the vanishing Ricci tensors and the corresponding vanishing scalar curvature. These tensors and the curvature have been used in Section II.

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